Building blocks of a black hole

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Abstract

What is the nature of the energy spectrum of a black hole ? The algebraic approach to black hole quantization requires the horizon area eigenvalues to be equally spaced. As stressed long ago by Mukhanov, such eigenvalues must be exponentially degenerate with respect to the area quantum number if one is to understand black hole entropy as reflecting degeneracy of the observable states. Here we construct the black hole stationary states by means of a pair of "creation operators" subject to a particular simple algebra, a slight generalization of that for a pair of harmonic oscillators. This algebra reproduces the main features of the algebraic approach, in particular the equally spaced area spectrum. We then prove rigorously that the n-th area eigenvalue is exactly 2^n-fold degenerate. Thus black hole entropy *qua* logarithm of the number of states for fixed horizon area is indeed proportional to that area.

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I. INTRODUCTION

Quantum gravity, the interplay of quantum theory with gravitation theory, remains one of the most interesting and challenging topics in theoretical physics today. Notwithstanding the extant theories [1–3] which purport to represent quantum gravity, there is as yet no clear and consistent picture of the subject. This is why a situation involving simultaneously strong gravitational fields as well as properties reminiscent of localized particles could shed light on the construction of the final version of quantum gravity. The simplicity of black holes makes them a salient candidate in this sense. Among the simplest questions that can be asked in quantum gravity is what is the nature of the energy spectrum of a black hole.

One of us noted early that the area of a black hole event horizon behaves somewhat like a classical adiabatic invariant [4] (see also refs. [5,6]). Ehrenfest's principle then suggests that the horizon area represents a quantum entity with a discrete spectrum [4,7–9]. Further, the fact that introducing a quantum particle into a Kerr-Newman black hole carries a minimal "cost" $\sim \hbar$ of area increase, which does not depend on the black hole parameters, suggests that the spacing between area eigenvalues is uniform [10,7–9]. The discrete nature of the eigenvalue spectrum for the horizon area is also supported by the loop quantum gravity (see Ashtekar and Krasnov in Ref. [2]), but this last theory suggests a rather complicated eigenvalue spacing. If the area spectrum is equispaced, the classical relation $A = 16\pi M^2$ ($c = G = 1$) for a Schwarzschild black hole implies the mass spectrum $M \sim \sqrt{\hbar n}$ for it, where $n = 1, 2, \cdots$. This type of spectrum has subsequently been obtained by many authors [11–14].

The adiabatic invariant approach mentioned is, of course, heuristic. Nowdays it is customary to draw conclusions about observable spectra from an algebra of observables. The loop quantum gravity [2] indeed seeks to determine the spectrum of horizon area, among others, from the algebra of geometric operators in the theory. A completely different approach [7–9,15] is to assume that each separate black hole state, which one assumes comes from a discrete set, is created from a "black hole vacuum" $|\mathrm{vac}\rangle$ by the operation of a certain unitary operator:

$$|njmqs\rangle = \hat{R}_{njmqs}|\mathrm{vac}\rangle. \tag{1}$$

Here $|njmqs\rangle$ is a one-black hole state with area a_n, angular momentum j (m represents the z component) and charge q (in units of the fundamental charge). The quantum number s distinguishes between different states with the same area, charge, angular momentum and its z-component The algebra of the various $\hat{R}$ operators together with the observables, horizon area $\hat{A}$, charge $\hat{Q}$ and angular momentum $\hat{\mathbf{J}}$, can be constructed from symmetry considerations together with the assumption that any commutator of two $\hat{R}$'s is linear in all other $\hat{R}$'s and $\hat{A}$, $\hat{Q}$ and $\hat{\mathbf{J}}$ (which linearity reflects the usual additivity of all these latter quantities) [8,9,15]. Such an algebra implies that the spectrum of $\hat{A}$ is equally spaced for all charges and angular momenta:

$$a_n = a_0 n; \qquad n = 1, 2, 3, \cdots, \tag{2}$$

where a_0 is a positive constant proportional to $\hbar$.

And where is the black hole entropy in all this ? Although the proportionality of black hole entropy to horizon area can be inferred solely by considering the black hole as a macroscopic system in thermal equilibrium with its surrounding [16], it is generally agreed today

that a crucial test of any proposed quantum gravity is its ability to recover the above proportionality from a count of "internal" black hole states. Such derivations of black hole entropy have been proffered in a number of string related contexts (principally for extreme black holes) [17], by exploiting the asymptotic conformal symmetry near the horizon [18], and in the loop quantum gravity [2]. In the algebraic approach on which we concentrate here, the horizon area eigenvalues are distributed rather sparsely. It was first observed by Mukhanov [12] (see also Ref. [19]) that the proportionality of black hole entropy to horizon area is conditional upon the degeneracy degree g_n, the number of states $|njmqs\rangle$ with a common area eigenvalue a_n, being given by $g_n = k^n$, where k is some integer greater than one.

Heuristic ways of understanding the exponential growth of degeneracy include the observation that a black hole can radiatively cascade from the n-th level to the ground state $n = 1$ by 2^n different paths depending on which area levels it passes through [20], or that it can be raised from the ground state to level n by steps in 2^n ways [19]. Another heuristic view is that the quantization law Eq. (2) suggests that the horizon may be regarded as parcelled into n patches of area a_0. If each can exists in k different quantum states, then the law $g = k^n$ is immediate [7]. Wheeler [21], Sorkin [22] and Kastrup [23] have proposed similar ideas.

The expectation of an exponential rise in g is not implicit in other approaches. Quantum loop gravity recovers the connection of the area spectrum with black hole entropy by predicting a very dense distribution of eigenvalues with little degeneracy, if any [2]. Approaches based on canonical quantum gravity sometimes predict *infinite* degeneracy of sparsely distributed eigenvalues [14]. An argument within the algebraic approach itself suggests that g would rise *at least as fast* as exponentially with n [7,9] if it could be assumed that the states $[\hat{R}_{njmqs}, \hat{R}_{1000s'}]|\mathrm{vac}\rangle$ with all allowed s' are independent. However, at least for the way we shall construct the $\hat{R}_{njmqs}$ in Sec. II, this last assumption cannot be maintained. Formal proof of the law $g_n = k^n$ has thus been lacking heretofore.

The purpose of the present paper is to show that the exponential law $g_n = k^n$ is indeed a consequence of the algebraic approach if one builds the R operators as products involving just two kinds of (noncommuting) operators, $\hat{a}$'s and $\hat{b}$'s. The required algebra of these "building blocks" is inferred in Sec. II from very general requirements, and is almost unique. Then a systematic procedure is developed in Sec. III for counting the number of distinct black hole states created out of the vacuum by the said operator products. It yields the required exponentially rising degeneracy. The assumptions made in building the algebra are summarized in Sec. IV, where possible extensions of the present ideas are mentioned.

II. THE ALGEBRA

A. Fundamental building blocks

We start from the intuitive assumption that there exist one-black hole states. The normalized vacuum state (no black hole) is denoted by $|\mathrm{vac}\rangle$, and states with nonzero area eigenvalue a_n are denoted by $|n, s\rangle$, where s is a generic symbol for any additional quantum numbers which distinguish between all states with common n. When the hole has no angular momentum or charge, we have $s = 0, 1, 2, \cdots, g_n - 1$, where g_n is the degeneracy of the said

states. We shall mostly phrase the discussion for this Schwarzschild black hole case, but our arguments can be generalized. As mentioned [7–9,15], operators $\hat{R}_{ns}$ are defined such that $|n,s\rangle = \hat{R}_{ns}|\text{vac}\rangle$. That is, $\hat{R}_{ns}$ creates a black hole with area a_n from the vacuum. One drawback of this scheme is that there an infinity of creation operators $\hat{R}_{ns}$. It would be nice to construct them from a small number of more fundamental "building blocks" out of which the whole algebra of the $\hat{R}$ operators follows. At a physical level such construction, if possible, should illuminate the inner structure of the black hole.

For simplicity we assume that $g_1 = 2$. Taking $g_1 = 3, 4, \cdots$ would change our main result only in some details. With $g_1 = 2$ the first area level has two independent quantum states, say $|1,0\rangle$ and $|1,1\rangle$. Let us try identifying the fundamental building blocks of the algebra with the $\hat{R}$ operators for these two states,

$$\hat{a} \equiv \hat{R}_{11} \quad \text{and} \quad \hat{b} \equiv \hat{R}_{12}, \tag{3}$$

so that

$$|1,0\rangle \equiv \hat{a}|\text{vac}\rangle \quad \text{and} \quad |1,1\rangle \equiv \hat{b}|\text{vac}\rangle. \tag{4}$$

By previous work [7–9,15] and Eq. (3), $\hat{a}$ and $\hat{b}$ should comply with

$$[\hat{A},\hat{a}] = a_0\hat{a} \quad \text{and} \quad [\hat{A},\hat{b}] = a_0\hat{b}, \tag{5}$$

where $\hat{A}$ is the positive semidefinite horizon area operator, and a_0 is a positive constant with the dimensions of area. Eqs. (5) are checked by operating with them on $|\text{vac}\rangle$ and taking into account that $\hat{A}|\text{vac}\rangle = 0$ because the vacuum contains no horizons. The commutators (5) are taken as axioms here; are they unique ?

Were one to add to the r.h.s. of the first of Eq.(5) a term involving $\hat{b}$, then $\hat{a}|\text{vac}\rangle$ would no longer be an eigenstate of $\hat{A}$, thus overturning the motive in defining $\hat{a}$. Similarly for an $\hat{a}$ dependent term in the r.h.s. of the second equation. But it does seem possible, from this point of view, to add to the r.h.s. in Eq. (5) terms of the form $h(\hat{A})$, provided the functions h vanish for zero argument. However, by making the redefinitions $\hat{a} \to \hat{a} + h_1(\hat{A})$ and $\hat{b} \to \hat{b} + h_2(\hat{A})$, we recover the original commutators (5). Hence for a pair of building blocks, Eq. (5) are the unique choice. We may immediately complement them with their conjugates,

$$[\hat{A},\hat{a}^\dagger] = -a_0\hat{a}^\dagger \quad \text{and} \quad [\hat{A},\hat{b}^\dagger] = -a_0\hat{b}^\dagger. \tag{6}$$

B. Completing the algebra

What are the algebraic relations among the elementary operators $\hat{a}$, $\hat{b}$, $\hat{b}^\dagger$ and $\hat{b}^\dagger$ by themselves ? One possible approach is that of Alekseev, Polychronakos and Smedbäck [11] who adopt the Cuntz algebra; in the case of two building blocks this has $\hat{a}^\dagger\hat{b} = \hat{b}^\dagger\hat{a} = 0$ and $\hat{a}^\dagger\hat{a} = \hat{b}^\dagger\hat{b} = 1$. This algebra yields—trivially—the exponentially rising degeneracy which is required by the black hole entropy law. We wish to stress here, however, the importance of stating the algebra of the building blocks exclusively in terms of commutators (as opposed to just products as in Cuntz's algebra). This is desirable, not only because the motivating

algebraic approach [7–9,15] is based exclusively on commutators, but also because of the well known connection between commutators and Poisson brackets in the classical-quantum correspondence. After all, such correspondence through the adiabatic invariance of horizon area served as motivation for our approach.

It is easy to verify from the Jacobi identity and Eq. (5)-(6) that the commutators $[\hat{a}^\dagger, \hat{b}]$, $[\hat{b}^\dagger, \hat{a}]$, $[\hat{a}^\dagger, \hat{a}]$ and $[\hat{b}^\dagger, \hat{b}]$ all commute with $\hat{A}$, but with no other elementary operator. Since the elementary operators here are $\hat{a}$, $\hat{b}$, $\hat{b}^\dagger$, $\hat{b}^\dagger$ and $\hat{A}$, the four mentioned commutators must be functions of $\hat{A}$ only:

$$\left[\hat{a}^\dagger, \hat{b}\right] = \left[\hat{b}^\dagger, \hat{a}\right]^\dagger = f(\hat{A}) \tag{7}$$

$$\left[\hat{a}^\dagger, \hat{a}\right] = \left[\hat{b}^\dagger, \hat{b}\right] = F(\hat{A}), \tag{8}$$

Here $f = \mathcal{R}(f) + \imath\mathcal{I}(f)$ may be complex, but F must be real because $[\hat{a}^\dagger, \hat{a}]$ is evidently hermitian. The assumed equality of the last two commutators requires comment. The spirit of the "no hair theorems" is that an uncharged and nonrotating black hole has only one observable degree of freedom, here taken as $\hat{A}$. In light of this it seems appropriate to demand $[\hat{a}^\dagger, \hat{a}] = [\hat{b}^\dagger, \hat{b}]$, since any asymmetry between the two commutators would provide an extra observable that distinguished between $\hat{a}$ and $\hat{b}$.

Keeping this in mind we see that it should make no difference physically if instead of the basis states (4) we use linear combinations of them, or equivalently, use linear combinations $\hat{a}$ and $\hat{b}$ to generate basis states. So suppose we replace the operators $\hat{a}$ and $\hat{b}$ by $\hat{a}'$ and $\hat{b}'$ according to

$$\hat{a}' = \hat{a}\cos\theta - \hat{b}\sin\theta \tag{9}$$

$$\hat{b}' = \hat{a}\sin\theta + \hat{b}\cos\theta \tag{10}$$

where θ is some real angle. Clearly the form of the commutators (5)-(6) is unaffected. However, we now have

$$\left[\hat{a}'^\dagger, \hat{b}'\right] = \mathcal{R}(f)\cos 2\theta + \imath\mathcal{I}(f) \tag{11}$$

$$\left[\hat{b}'^\dagger, \hat{a}'\right] = \mathcal{R}(f)\cos 2\theta - \imath\mathcal{I}(f) \tag{12}$$

$$\left[\hat{a}'^\dagger, \hat{a}'\right] = F - \mathcal{R}(f)\sin 2\theta. \tag{13}$$

$$\left[\hat{b}'^\dagger, \hat{b}'\right] = F + \mathcal{R}(f)\sin 2\theta \tag{14}$$

It is apparent that if $\mathcal{R}(f) \neq 0$, the form of the algebra varies with θ contrary to the principle mentioned. To eliminate the undesirable dependence on $\mathcal{R}(f)$, we must demand that $\mathcal{R}(f) = 0$.

This understood, let us shift the phase of $\hat{a}$ by multiplying it by $\imath$. Clearly this is an allowed redefinition of $\hat{a}$ which should have no physical consequences. Indeed, it leaves the form of Eqs. (5), (6), (13) and (14) unaffected. However, it interchanges the roles of $\mathcal{R}(f)$ and $\mathcal{I}(f)$ in Eqs. (11) and (12). Thus if we afterwards carry out a rotation of the form (9)-(10), we obvioulsy find that whenever $\mathcal{I}(f) \neq 0$, the algebra depends on θ. The already explained logic thus forces us to set $\mathcal{I}(f) = 0$ as well. Overall we must have $f = 0$.

Let us operate with Eq. (6) on $|\mathrm{vac}\rangle$. We find $\hat{A}\hat{a}^\dagger|\mathrm{vac}\rangle = -a_0\hat{a}^\dagger|\mathrm{vac}\rangle$. But $\hat{A}$ is a positive definite operator, so this can only mean that $\hat{a}^\dagger$ anhilates the vacuum. A similar conclusion applies to $\hat{b}^\dagger$. It follows from Eqs. (7) and (8) that

$$\langle \mathrm{vac}|\hat{a}^\dagger\hat{b}|\mathrm{vac}\rangle = \langle \mathrm{vac}|\hat{b}^\dagger\hat{a}|\mathrm{vac}\rangle = 0 \tag{15}$$

$$\langle \mathrm{vac}|\hat{a}^\dagger\hat{a}|\mathrm{vac}\rangle = \langle \mathrm{vac}|\hat{b}^\dagger\hat{b}|\mathrm{vac}\rangle = F(0). \tag{16}$$

The first equation tells us that the basis states $\hat{a}|\mathrm{vac}\rangle$ and $\hat{b}|\mathrm{vac}\rangle$ are orthogonal, which is convenient in what follows. To normalize them we require that F be such that $F(0) = 1$.

It will be convenient below to restrict attention to a linear $F(\hat{A})$; but as we explain in Appendix A, our results about degeneracy remain valid for almost any choice of function provided $F(x) > 0$ for $x > 0$. In light of these remarks we may focus on the algebra with the form

$$\left[\hat{a}^\dagger, \hat{b}\right] = \left[\hat{b}^\dagger, \hat{a}\right] = 0. \tag{17}$$

$$\left[\hat{a}^\dagger, \hat{a}\right] = \left[\hat{b}^\dagger, \hat{b}\right] = 1 + \alpha\hat{A} \equiv 1 + w\hat{N}, \tag{18}$$

where α is an unknown parameter with dimensions of 1/(area), $\hat{N} \equiv \hat{A}/a_0$ is the dimensionless area operator and $w \equiv \alpha a_0$. We shall prove in Sec. III A that necessarily $w > 0$. As a final point here we remark that there is no reason for trying to express $[\hat{a}, \hat{b}]$ in terms of the other basic operators in the algebra; $[\hat{a}, \hat{b}]$ is to be regarded as a new (certainly non-vanishing) operator. The relevance of the algebra (17)-(18) for the problem at hand was first appreciated in conversations of one of us with V. Mukhanov. By contrast with the case of Cuntz's algebra, the establishment of the degeneracy of the area levels for the algebra (17)-(18) is complicated, and requires the special methods developed in Sec. III.

III. DEGENERACY OF THE AREA LEVELS

A. Degeneracy of the $n = 2$ area level

As mentioned, the first area level, $n = 1$ is doubly degenerate. What is the degeneracy of the $n = 2$ states ? By combining Eq. (5) with the Jacobi identity we find that

$$[\hat{A}, \hat{a}\hat{a}] = 2a_0\hat{a}\hat{a}; \quad [\hat{A}, \hat{b}\hat{b}] = 2a_0\hat{b}\hat{b}; \quad [\hat{A}, \hat{a}\hat{b}] = 2a_0\hat{a}\hat{b}; \quad [\hat{A}, \hat{b}\hat{a}] = 2a_0\hat{b}\hat{a}. \tag{19}$$

In view of these, let us define four states while introducing a new symbol for states:

$$|00\rangle\!\rangle \equiv \hat{a}\hat{a}|\mathrm{vac}\rangle,\ |01\rangle\!\rangle \equiv \hat{a}\hat{b}|\mathrm{vac}\rangle,\ |10\rangle\!\rangle \equiv \hat{b}\hat{a}|\mathrm{vac}\rangle \quad \text{and} \quad |11\rangle\!\rangle \equiv \hat{b}\hat{b}|\mathrm{vac}\rangle. \tag{20}$$

In a ket of type $|\ \rangle\!\rangle$ a "0" is created by the action of operator $\hat{a}$ and a "1" by that of $\hat{b}$. Operating on $|\mathrm{vac}\rangle$ with Eq. (19) we find that the above four states are states with area $2a_0$ corresponding to $n = 2$. Note that the string of "0" and "1"'s in a state $|\ \rangle\!\rangle$ is the binary representation of s in our original notation $|2, s\rangle$ with $s = 0, \cdots, 3$.

All states with $n = 2$ must be superpositions of the four states in Eq.(20) since there are no other two-operator products, and it is easy to see, by extending the calculation entailed in Eq. (19), that three-operator product states, like $\hat{a}\hat{b}\hat{a}|\mathrm{vac}\rangle$ correspond rather to $n = 3$,

and correspondingly larger n for products of n operators. We now prove that the four states are linearly independent.

Using Eqs. (18) and (17) one finds that

$$\begin{aligned}
\langle\langle 00|00\rangle\rangle &= \langle \text{vac}|\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}|\text{vac}\rangle = \langle \text{vac}|\hat{a}^\dagger(1+w\hat{N})\hat{a}|\text{vac}\rangle + \langle \text{vac}|\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}|\text{vac}\rangle = 2+w \\
\langle\langle 10|00\rangle\rangle &= \langle \text{vac}|\hat{a}^\dagger\hat{b}^\dagger\hat{a}\hat{a}|\text{vac}\rangle = \langle \text{vac}|\hat{a}^\dagger\hat{a}\hat{a}\hat{b}^\dagger|\text{vac}\rangle = 0 \\
\langle\langle 10|01\rangle\rangle &= \langle \text{vac}|\hat{a}^\dagger\hat{b}^\dagger\hat{a}\hat{b}|\text{vac}\rangle = \langle \text{vac}|\hat{a}^\dagger\hat{a}\hat{b}^\dagger\hat{b}|\text{vac}\rangle = 1 \\
\langle\langle 01|01\rangle\rangle &= \langle \text{vac}|\hat{b}^\dagger\hat{a}^\dagger\hat{a}\hat{b}|\text{vac}\rangle = \langle \text{vac}|\hat{b}^\dagger(1+w\hat{N})\hat{b}|\text{vac}\rangle = 1+w.
\end{aligned} \tag{21}$$

By utilizing the symmetry under $\hat{a} \leftrightarrow \hat{b}$ one can calculate the rest of the scalar products. Summarizing the scalar products in matrix form gives

$$\begin{pmatrix} \langle\langle 00|00\rangle\rangle & \langle\langle 00|01\rangle\rangle & \langle\langle 00|10\rangle\rangle & \langle\langle 00|11\rangle\rangle \\ \langle\langle 01|00\rangle\rangle & \langle\langle 01|01\rangle\rangle & \langle\langle 01|10\rangle\rangle & \langle\langle 01|11\rangle\rangle \\ \langle\langle 10|00\rangle\rangle & \langle\langle 10|01\rangle\rangle & \langle\langle 10|10\rangle\rangle & \langle\langle 10|11\rangle\rangle \\ \langle\langle 11|00\rangle\rangle & \langle\langle 11|01\rangle\rangle & \langle\langle 11|10\rangle\rangle & \langle\langle 11|11\rangle\rangle \end{pmatrix} = \begin{pmatrix} 2+w & 0 & 0 & 0 \\ 0 & 1+w & 1 & 0 \\ 0 & 1 & 1+w & 0 \\ 0 & 0 & 0 & 2+w \end{pmatrix}. \tag{22}$$

We now show that $w > 0$. Define $|\psi\rangle \equiv |01\rangle\rangle - |01\rangle\rangle$. We have

$$\langle\psi|\psi\rangle = \langle\langle 01|01\rangle\rangle + \langle\langle 10|10\rangle\rangle - 2\langle\langle 01|10\rangle\rangle = 2w \tag{23}$$

Of course a minimum requirement is that the norm of a nontrivial state should be positive. Hence $w > 0$.

The determinant of the matrix in Eq. (22) is $w^4 + 6w^3 + 12w^2 + 8w$. Now were the four states in question linearly dependent, the above determinant would have to vanish (a column being a linear combination of the other three). But $w > 0$, so the four states are linearly independent. This means that the degeneracy of the second area level is $g_2 = 4 = 2^2$.

This is a good point to indicate why our choice of $F(\hat{A}) = 1 + w\hat{A}/a_0$ does not restrict the generality of the conclusions drawn here and below. With general $F(\hat{A})$ satisfying $F(0) = 1$ a repetition of the above calculation shows the nonvanishing entries in the matrix (22) are replaced according to $2 + w \to 1 + F(a_0)$, $1 + w \to F(a_0)$ with the unit entry replaced by $F(0) = 1$. The positivity of the norm $\langle\psi|\psi\rangle$ would tell us that $F(a_0) > 1$. The determinant is replaced by $(1 + F(a_0))^3(F(a_0) - 1)$ which is evidently positive. We again conclude that the four states are linearly independent.

B. Degeneracy of the $n = 3$ and $n = 4$ area levels

For $n = 3$ the eight states are $|3,0\rangle = |000\rangle\rangle = \hat{a}\hat{a}\hat{a}|\text{vac}\rangle$ and analogously $|3,1\rangle = |001\rangle\rangle$, $|3,2\rangle = |010\rangle\rangle$, $|3,3\rangle = |011\rangle\rangle$, $|3,4\rangle = |100\rangle\rangle$, $|3,5\rangle = |101\rangle\rangle$, $|3,6\rangle = |110\rangle\rangle$ and $|3,7\rangle = |111\rangle\rangle$. Note again that the sequence of 3-bit "0" and "1"'s is the binary representation of s in the $|\ \rangle$ form of the ket, while the index $n = 3$ is connected with the fact that the binary representation is a 3-bit one. The 8×8 matrix of scalar products of the eight states $|3, s\rangle$ has been calculated by means of a dedicated program in *Mathematica* which implements the operator algebra of Eqs. (18) and (17). Thus to calculate $\langle\langle 001|010\rangle\rangle = \langle \text{vac}|\hat{b}^\dagger\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{b}\hat{a}|\text{vac}\rangle$ one commutes all the $\hat{a}^\dagger$ and the $b^\dagger$ to the right until they reach $|\text{vac}\rangle$ and anhilate it. The constants produced by the commutations add up to $3w + 2$. The full matrix is

$$\begin{pmatrix} 3\left(2+3w+w^2\right) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2+5w+3w^2 & 2+3w & 0 & 2+w & 0 & 0 & 0 \\ 0 & 2+3w & 2+3w+2w^2 & 0 & 2+3w+w^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2+5w+2w^2 & 0 & 2+3w+w^2 & 2+w & 0 \\ 0 & 2+w & 2+3w+w^2 & 0 & 2+5w+2w^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2+3w+w^2 & 0 & 2+3w+2w^2 & 2+3w & 0 \\ 0 & 0 & 0 & 2+w & 0 & 2+3w & 2+5w+3w^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3\left(2+3w+w^2\right) \end{pmatrix} \tag{24}$$

and its determinant is $46656\,w^6 + 326592\,w^7 + 1014768\,w^8 + 1842912\,w^9 + 2166588\,w^{10} + 1723356\,w^{11} + 939681\,w^{12} + 347004\,w^{13} + 83106\,w^{14} + 11664\,w^{15} + 729\,w^{16}$, obviously nonvanishing for $w > 0$. Hence the eight $n = 3$ states are linearly independent and the degeneracy of the third area level is thus $g_3 = 8 = 2^3$.

For $n = 4$ the states are formed by operating a string of four $\hat{a}$'s and $\hat{b}$'s on $|\text{vac}\rangle$. Each of these sixteen states $|4, s\rangle$ with $s = 0, 1, \cdots, 15$ corresponds to a state of the form $|\cdots\rangle\!\rangle$ where the 4-bit binary number equivalent to s reflects the four operators product used in its construction in accordance with the equivalence $0 \Leftrightarrow \hat{a}$ and $1 \Leftrightarrow \hat{b}$. One can calculate the scalar products between pairs of states as before; we shall forego the display of the 16×16 matrix, or its determinant which is also positive for $w > 0$. Therefore, the sixteen $n = 4$ states are linearly independent and the degeneracy of the fourth area level is $g_4 = 16 = 2^4$.

The pattern is now clear and we proceed to prove analytically that for a general n area level the degeneracy is $g_n = 2^n$.

C. Proof of 2^n-fold degeneracy of the n-th area level

We first define 2^n states with area eigenvalue na_0 as follows:

$$|x_1 x_2 \cdots x_n\rangle\!\rangle \equiv \hat{x}_1 \hat{x}_2 \cdots \hat{x}_n |\text{vac}\rangle \tag{25}$$

where $x_i = 0$ or 1 and correspondingly $\hat{x}_i$ is either $\hat{a}$ or $\hat{b}$ ($i = 1, 2, .., n$). Therefore, there are exactly 2^n states.

Theorem: All the 2^n states defined in Eq. (25) are linearly independent.

This theorem implies that the degeneracy of the nth area level is $g_n = 2^n$. In order to prove the theorem, we first define an operator $\hat{Z}$ which we denote "quasi-charge",

$$\hat{Z}|x_1 x_2 \cdots x_n\rangle\!\rangle \equiv \left(\sum_{i=1}^{n} x_i\right) |x_1 x_2 \cdots x_n\rangle\!\rangle. \tag{26}$$

Since $x_i = 0$ or 1 the sum $z \equiv \sum_{i=1}^{n} x_i$ counts the number of times that $\hat{b}$ appears in the construction of $|x_1 x_2 \cdots x_n\rangle\!\rangle$.

Lemma: States of like area but different quasi-charge are orthogonal to each other.

Proof: We use induction. For $n = 2$ the result is clear from Eq. (22). Assuming now that it is correct for $n - 1$, we shall prove it for n. Let $|x_1 x_2 \cdots x_n\rangle\!\rangle$ and $|x'_1 x'_2 \cdots x'_n\rangle\!\rangle$ have different quasi-charges and consider the following two cases:

i) $x_1 = x'_1$. In this case, by assumption, the state $|x_2 \cdots x_n\rangle\!\rangle$ is orthogonal to $|x'_2 \cdots x'_n\rangle\!\rangle$ since they must have different quasi-charges. Hence,

$$\begin{aligned} \langle\!\langle x_1 x_2 \cdots x_n | x'_1 x'_2 \cdots x'_n \rangle\!\rangle = \langle\!\langle x_2 \cdots x_n | \hat{x}_1^\dagger \hat{x}'_1 | x'_2 \cdots x'_n \rangle\!\rangle \\ = \langle\!\langle x_2 \cdots x_n | \hat{x}'_1 \hat{x}_1^\dagger | x'_2 \cdots x'_n \rangle\!\rangle + [1 + (n-1)w] \langle\!\langle x_2 \cdots x_n | x'_2 \cdots x'_n \rangle\!\rangle = 0, \end{aligned} \tag{27}$$

because the state $(\hat{x}'_1 \hat{x}_1^\dagger)|x'_2 \cdots x'_n\rangle\!\rangle$ can evidently be written as a superposition of states with the same quasi-charge (and area) as the state $|x'_2 \cdots x'_n\rangle\!\rangle$.

ii) $x_1 \neq x'_1$. Without loss of generality, we shall assume that $\hat{x}'_1 = \hat{a}$ and $\hat{x}_1 = \hat{b}$. If we denote the quasi-charge of $|x_2 \cdots x_n\rangle\!\rangle$ by z, then, the quasi-charge of $|x'_2 \cdots x'_n\rangle\!\rangle$ must be *different* from $z+1$. Now, when the operator $\hat{b}^\dagger \hat{a}$ act on the state $|x'_2 \cdots x'_n\rangle\!\rangle$ it preserves the state's area but decreases its quasi-charge by one. Thus, the state $\hat{b}^\dagger \hat{a}|x'_2 \cdots x'_n\rangle\!\rangle$ can be written as a superposition of states with area $(n-1)a_0$ and with a quasi-charge which is different from z. By our assumption $\hat{b}^\dagger \hat{a}|x'_2 \cdots x'_n\rangle\!\rangle$ is thus orthogonal to $|x_2 \cdots x_n\rangle\!\rangle$ and hence

$$\langle\!\langle x_2 \cdots x_n | \hat{b}^\dagger \hat{a} | x'_2 \cdots x'_n \rangle\!\rangle = \langle\!\langle x_1 x_2 \cdots x_n | x'_1 x'_2 \cdots x'_n \rangle\!\rangle = 0. \tag{28}$$

This proves the case n; hence by induction states with different quasi-charge are orthogonal.

The 2^n states defined in Eq. (25) can be divided into $n+1$ groups, each characterized by the quasi-charge of its states: $z = 0, 1, \cdots, n$. Thus, the number of states in the z group is $\binom{n}{z}$ and the total number of states with area na_0 is $\sum_{z=0}^{n} \binom{n}{z} = 2^n$. Since states with different z are orthogonal, it is enough to prove that the $\binom{n}{z}$ states in each z group are all independent. In the following, states $|x_1 x_2 \cdots x_n\rangle\!\rangle$ and $|x'_1 x'_2 \cdots x'_n\rangle\!\rangle$ with the same z will be denoted by $|n, z, l\rangle$ and $|n, z, l'\rangle$, respectively, where $l, l' = 1, 2, \cdots, \binom{n}{z}$.

In order to prove the theorem, it is necessary to know the form of the scalar product between two general states with the same quasi-charge z. In Appendix A it is shown that

$$\langle n, z, l | n, z, l' \rangle = \sum_{\tilde{p} \in \mathcal{P}_{l,l'}} h(\tilde{p}), \tag{29}$$

where $\mathcal{P}_{l,l'}$ is the set of $z!(n-z)!$ permutations (a subset of all the $n!$ permutations constituting the symmetric, or permutation, group over n objects) that take string $x'_1 x'_2 \cdots x'_n$ representing $|n, z, l\rangle$ into $x'_1 x'_2 \cdots x'_n$ representing $|n, z, l'\rangle$. The function $h(\tilde{p})$ is a specific one-to-one function that maps each particular permutation $\tilde{p}$ to a positive number.

We shall prove by contradiction that the determinant of the matrix $M^{(n,z)}$ with components $M^{(n,z)}_{ll'} \equiv \langle n, z, l | n, z, l' \rangle$ is nonvanishing. Let us assume otherwise. Then there should be at least one $\binom{n}{z}$ dimensional vector $\vec{C} \neq 0$ which satisfies $M^{(n,z)} \vec{C} = 0$. This implies that

$$\sum_{l'=1}^{\binom{n}{z}} M^{(n,z)}_{ll'} c_{l'} = \sum_{l'=1}^{\binom{n}{z}} \sum_{\tilde{p} \in \mathcal{P}_{l,l'}} h(\tilde{p}) c_{l'} = 0, \tag{30}$$

where not all the $c_{l'}$ are zero. Since each group $\mathcal{P}_{l,l'}$ contains exactly $z!(n-z)!$ permutations, the sums in Eq. (30) contains $z!(n-z)! \cdot \binom{n}{z} = n!$ terms. Furthermore,

$$\mathcal{P}_{l,l'} \cap \mathcal{P}_{l,l''} = \mathcal{P}_{l',l} \cap \mathcal{P}_{l'',l} = \{\emptyset\} \tag{31}$$

for $l' \neq l''$ because a permutation $\tilde{p} \in \mathcal{P}_{l,l'}$ takes the state $|n, z, l\rangle$ into the state $|n, z, l'\rangle$, but cannot take $|n, z, l\rangle$ into $|n, z, l''\rangle$. Note also that

$$\mathcal{P} = \bigcup_{l'=1}^{\binom{n}{z}} \mathcal{P}_{l',l} = \bigcup_{l'=1}^{\binom{n}{z}} \mathcal{P}_{l,l'} \qquad \forall \quad 1 \leq l \leq \binom{n}{z}, \tag{32}$$

where $\mathcal{P}$ is the symmetric group over n objects. Eq. (31) and Eq. (32) will be very helpful in the following definitions.

Let us define the matrices G_l ($l = 1, 2, \cdots, \binom{n}{z}$), each of dimension $z!(n-z)! \times \binom{n}{z}$. The k-th row of G_l is the string of $z!(n-z)!$ randomly ordered distinct numbers $h(\tilde{p})$ with $\tilde{p} \in \mathcal{P}_{k,l}$ [as mentioned, $\mathcal{P}_{k,l}$ contains $z!(n-z)!$ permutations]. Note that Eqs. (31)- (32) imply that each matrix G_l contains exactly all the $n!$ terms $h(\tilde{p})$ with $\tilde{p} \in \mathcal{P}$.

We now construct the matrix $H_1 \equiv G_1 \cup G_2 \cup \cdots \cup G_{\binom{n}{z}}$, of dimension $n! \times \binom{n}{z}$ by taking its columns to be the columns of all the G_l's in the given order. By enlarging the $\binom{n}{z}$ dimensional vector $\vec{C}$ into the $n!$ dimensional vector

$$\vec{C}_{\text{enlarged}} = (\underbrace{c_1, c_1, \cdots, c_1}_{z!(n-z)!}, \underbrace{c_2, c_2, \cdots, c_2}_{z!(n-z)!}, \cdots, \underbrace{c_{\binom{n}{z}}, c_{\binom{n}{z}}, \cdots, c_{\binom{n}{z}}}_{z!(n-z)!}), \tag{33}$$

one can show that Eq. (30) is equivalent to $H_1\vec{C}_{\text{enlarged}} = 0$.

The matrix $G_l^{(m)}$, where $m = 1, 2, \cdots, z!(n-z)!$, is obtained after performing m cyclic permutations to the columns of G_l. Thus for $m = 1$ the second column is replaced by the first, the third by the second, etc. For $m = 2$ the first is replaced by the third, etc. Hence, all the $z!(n-z)!$ matrices $H_m \equiv G_1^{(m-1)} \cup G_2^{(m-1)} \cup \cdots \cup G_{\binom{n}{z}}^{(m-1)}$ satisfy $H_m\vec{C}_{\text{enlarged}} = 0$. Finally, the square matrix H of dimension $n! \times n!$ is defined such that its first $\binom{n}{z}$ rows are given by H_1, the second $\binom{n}{z}$ rows by H_2 and so on. Therefore, it is clear that also $H\vec{C}_{\text{enlarged}} = 0$.

In each row and each column of H *all* the $n!$ numbers $h(\tilde{p})$ appear. Hence, by writing out the set $\{h(\tilde{p})|\tilde{p} \in \mathcal{P}\}$ as $h_1, h_2, \cdots, h_{n!}$, we find that

$$H = \begin{pmatrix} h_1 & h_2 & h_3 & \cdot & \cdot & \cdot & h_{n!-1} & h_{n!} \\ h_{n!} & h_1 & h_2 & \cdot & \cdot & \cdot & h_{n!-2} & h_{n!-1} \\ h_{n!-1} & h_{n!} & h_1 & \cdot & \cdot & \cdot & h_{n!-3} & h_{n!-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ h_2 & h_3 & h_4 & \cdot & \cdot & \cdot & h_{n!} & h_1 \end{pmatrix} \tag{34}$$

where we have rearranged the rows in H (changing the orders of the rows in H does not affect the equation $H\vec{C}_{\text{enlarged}} = 0$).

Let us now recall the $n!$-th roots of unity:

$$\varepsilon_m = \exp\left(i\frac{2\pi}{n!}m\right); \qquad m = 1, 2, \cdots, n! \tag{35}$$

It may be checked that the eigenvectors of H are

$$\vec{e}_k \equiv ({\varepsilon_1}^k, {\varepsilon_2}^k, \cdots, {\varepsilon_m}^k, \cdots, {\varepsilon_{n!}}^k); \qquad k = 1, 2, \cdots, n! \tag{36}$$

with corresponding eigenvalues

$$\lambda_k = h_1 + h_2{\varepsilon_1}^k + h_3{\varepsilon_2}^k + \cdots + h_{n!}{\varepsilon_{n!-1}}^k. \tag{37}$$

Because ${\varepsilon_1}^{n!} = 1$, and the h_m are positive, $\lambda_{n!} > 0$. It can be shown (Appendix B) that $\lambda_k \neq 0$ also for $k < n!$. Thus the determinant of H is *not* zero. This contradicts our tentative assumption that there exists a vector $\vec{C} \neq 0$ such that $M^{(n,z)}\vec{C} = 0$ because this is equivalent to assuming that $H\vec{C}_{\text{enlarged}} = 0$. Thus the matrix $M^{(n,z)}$ must have nonvanishing determinant, which proves that all the 2^n states defined in Eq. (25) are linearly independent, as claimed. In the last paragraph of Appendix A we explain why this result remains unchanged if our choice $F(\hat{A}) = 1 + w\hat{A}/a_0$ is replaced by almost any other function which is positive for positive argument.

IV. SUMMARY AND CONCLUSIONS

The equally spaced area spectrum of a stationary black hole raises the question of the degeneracy of area states. An old argument by Mukhanov [12] suggests that the degeneracy should rise exponentially with the area quantum number n if the black hole entropy is to be understood as the logarithm of the number of "microstates" per state with definite observable parameters, e.g. area, charge, etc. What algebra of operators would be conducive to such behavior ? We have here assumed that the generic black hole state is created by operating on the vacuum with a string of "raising" operators of just two kinds, $\hat{a}$ and $\hat{b}$ (building blocks). Assuming the commutator of either $\hat{a}$ or $\hat{b}$ with the area operator $\hat{A}$ is proportional to itself, this construction explains the equispaced area spectrum. By further assuming that the algebra of the $\hat{a}$ and $\hat{b}$ operators and their adjoints is exclusively defined by commutators, and that no physical consequences follow when $\hat{a}$ and $\hat{b}$ are redefined as linear combinations of themselves, we have singled out the algebra, and we have shown that it leads to the degeneracy law $g_n = 2^n$ which is of the type needed to explain the black hole entropy as a reflection of area eigenstate degeneracy.

The above described arguments do not depend on the exact nature of the stationary black hole: spherical or rotating, neutral or electrically charged. It would evidently be interesting to associate with the $\hat{a}$ and $\hat{b}$ operators some angular momentum and/or charge, and so build up specifically Reissner-Nordström and Kerr black hole states. We know that some degeneracy accrues to systems with angular momentum by virtue of rotational symmetry: states with definite area na_0, squared angular momentum $j(j+1)\hbar^2$ and charge q should comprise substates differing only by the z-component of angular momentum. Thus the black hole degeneracy might be expected to depend not only on the area quantum number n, but also on the angular momentum j. However, according to the first law of thermodynamics, the black hole entropy is a function of the horizon area alone [16] and, therefore, so should the degeneracy. This implies that the spectrum of the horizon area of a black hole must depend on all of n, j and q. This argument is consistent with the result from canonical quantum gravity obtained by Barvinsky, Das and Kunstatter [24] for the area spectrum of charged black holes and gives a further motivation for our algebraic approach.

V. ACKNOWLEDGMENTS

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APPENDIX A: SCALAR PRODUCT OF TWO GENERIC STATES

Let $|x_1 x_2 \cdots x_n\rangle\!\rangle$ and $|x'_1 x'_2 \cdots x'_n\rangle\!\rangle$ be two states with the same area (later we shall assume the same z also). Their scalar product is

$$\langle\!\langle x_1 x_2 \ \cdots x_n | x'_1 x'_2 \ \cdots \ x'_n\rangle\!\rangle = \langle \mathrm{vac}|\hat{x}_n^\dagger \hat{x}_{n-1}^\dagger \cdots \hat{x}_1^\dagger \hat{x}'_1 \hat{x}'_2 \cdots \hat{x}'_n |\mathrm{vac}\rangle, \tag{A1}$$

which we rewrite by succesively moving $\hat{x}_1^\dagger$ all the way to the right using Eqs. (18)-(17):

$$\begin{aligned}
&= \delta_{x_1,x'_1}[1+(n-1)w]\langle \mathrm{vac}|\hat{x}_n^\dagger \cdots \hat{x}_2^\dagger \hat{x}'_2 \cdots \hat{x}'_n|\mathrm{vac}\rangle + \langle \mathrm{vac}|\hat{x}_n^\dagger \cdots \hat{x}_2^\dagger \hat{x}'_1 \hat{x}_1^\dagger \hat{x}'_2 \cdots \hat{x}'_n|\mathrm{vac}\rangle \\
&= \delta_{x_1,x'_1}[1+(n-1)w]\langle\!\langle x_2 \cdots x_n|x'_2 \cdots x'_n\rangle\!\rangle + \delta_{x_1,x'_2}[1+(n-2)w]\langle\!\langle x_2 \cdots x_n|x'_1 x'_3 \cdots x'_n\rangle\!\rangle \\
&+ \cdots + \delta_{x_1,x'_n}\langle\!\langle x_2 \cdots x_n|x'_1 \cdots x'_{n-1}\rangle\!\rangle.
\end{aligned} \tag{A2}$$

We have used the fact that $x_1^\dagger|\mathrm{vac}\rangle = 0$. Now we move $\hat{x}_2^\dagger$ all the way to the right

$$\begin{aligned}
&\langle\!\langle x_1 x_2 \cdots x_n|x'_1 x'_2 \cdots x'_n\rangle\!\rangle = \delta_{x_1,x'_1}[1+(n-1)w]\Big\{\delta_{x_2,x'_2}[1+(n-2)w]\langle\!\langle x_3 \cdots x_n|x'_3 \cdots x'_n\rangle\!\rangle \\
&+ \delta_{x_2,x'_3}[1+(n-3)w]\langle\!\langle x_3 \cdots x_n|x'_2 x'_4 x'_5 \cdots x'_n\rangle\!\rangle + \cdots + \delta_{x_2,x'_n}\langle\!\langle x_3 \cdots x_n|x'_2 \cdots x'_{n-1}\rangle\!\rangle\Big\} \\
&+ \delta_{x_1,x'_2}[1+(n-2)w]\Big\{\delta_{x_2,x'_1}[1+(n-2)w]\langle\!\langle x_3 \cdots x_n|x'_3 \cdots x'_n\rangle\!\rangle \\
&+ \delta_{x_2,x'_3}[1+(n-3)w]\langle\!\langle x_3 \cdots x_n|x'_1 x'_4 x'_5 \cdots x'_n\rangle\!\rangle + \cdots + \delta_{x_2,x'_n}\langle\!\langle x_3 \cdots x_n|x'_1 x'_3 x'_4 \cdots x'_{n-1}\rangle\!\rangle\Big\} \\
&+ \cdots + \delta_{x_1,x'_n}\Big\{\delta_{x_2,x'_1}[1+(n-2)w]\langle\!\langle x_3 \cdots x_n|x'_2 \cdots x'_{n-1}\rangle\!\rangle \\
&+ \delta_{x_2,x'_2}[1+(n-3)w]\langle\!\langle x_3 \cdots x_n|x'_1 x'_3 x'_4 \cdots x'_{n-1}\rangle\!\rangle + \cdots + \delta_{x_2,x'_{n-1}}\langle\!\langle x_3 \cdots x_n|x'_1 x'_2 \cdots x'_{n-2}\rangle\!\rangle\Big\}.
\end{aligned} \tag{A3}$$

Thus generically the scalar product is a sum of many (actually $n!$) terms. One example is

$$[1+(n-1)w][1+(n-2)w]\cdots[1+(n-n)w]\delta_{x_1,x'_1}\delta_{x_2,x'_2}\cdots\delta_{x_n,x'_n}.$$

obtained by converting by the aforesaid means the first term within the first curly brackets in Eq. (A3). Other examples include the term

$$[1+(n-1)w][1+(n-3)w][1+(n-3)w][1+(n-4)w]\cdots[1+(n-n)w]\delta_{x_1,x'_1}\delta_{x_2,x'_3}\delta_{x_3,x'_2}\cdots\delta_{x_n,x'_n}$$

resulting from expansion of the second term within the same brackets, and the term

$$[1+(n-1)w][1+(n-2)w]\cdots[1+(n-n)w]\delta_{x_1,x'_1}\delta_{x_2,x'_2}\cdots\delta_{x_n,x'_n}.$$

coming from the last term within the last curly brackets of Eq. (A3). Summing up, the scalar product has the following form:

$$\langle\langle x_1 x_2 \cdots x_n | x_1' x_2' \ \cdots \ x_n' \rangle\rangle$$
$$= \sum_{\tilde{p}\in\mathcal{P}} [1+(n-i_1)w][1+(n-i_2)w]\cdots[1+(n-i_n)w]\delta_{x_1,x'_{p_1}}\delta_{x_2,x'_{p_2}}\cdots\delta_{x_n,x'_{p_n}} \ , \quad \text{(A4)}$$

where $\mathcal{P}$ is the (symmetric) group of all $n!$ permutations $\tilde{p} \equiv (p_1, p_2, \cdots, p_n)$ of the objects labelled by $1, 2, \cdots, n$ and $i_1, i_2, \cdots, i_n$ are n integers satisfying $1 \le i_1 \le n, \ 2 \le i_2 \le n, \ \cdots, n-1 \le i_{n-1} \le n, \ i_n = n$. Hence, there are exactly $n!$ sets of $i_1, i_2, \cdots, i_n$ and each permutation $\tilde{p}$ can be regarded as associated with a single set $i_1(\tilde{p}), i_2(\tilde{p}), \cdots, i_n(\tilde{p})$.

Eq. (A4) supplies an alternative proof of the lemma of section III: the scalar product of two states with different quasi-charge must be zero. This is because $\delta_{x_1,x'_{p_1}}\delta_{x_2,x'_{p_2}}\cdots\delta_{x_n,x'_{p_n}} = 0$ for all $\tilde{p}$. Therefore, we shall restrict ourselves to states with a fixed quasi-charge z.

As mentioned in Sec. III C, there are $\binom{n}{z}$ states with the same z, we shall denote them by $|n, z, l\rangle\rangle \equiv |x_1 x_2 \cdots x_n\rangle\rangle$ (or $|n, z, l'\rangle\rangle \equiv |x_1' x_2' \cdots x_n'\rangle\rangle$), where $l, l' = 1, 2, \cdots, \binom{n}{z}$. Furthermore, the product $\delta_{x_1,x'_{p_1}}\delta_{x_2,x'_{p_2}}\cdots\delta_{x_n,x'_{p_n}}$ is not zero for exactly $z!(n-z)!$ permutations. Thus, we shall denote by $\mathcal{P}_{l,l'}$ the group of $z!(n-z)!$ permutations that contribute to the scalar product of $|n, z, l\rangle\rangle$ with $|n, z, l'\rangle\rangle$. Using these notation, we can write the scalar product of two states in a compact form:

$$\langle\langle n, z, l | n, z, l' \rangle\rangle = \sum_{\tilde{p}\in\mathcal{P}_{l,l'}} \prod_{k=1}^{n} [1 + (n - i_k(\tilde{p}))w] \equiv \sum_{\tilde{p}\in\mathcal{P}_{l,l'}} h(\tilde{p}). \quad \text{(A5)}$$

Notice that all $h(\tilde{p})$ are positive and different.

In the paper, the above explicit expression for $h(\tilde{p})$ is not used. Therefore, the choice of $F(\hat{A}) = 1 + w\hat{A}/a_0$ does not restrict the generality of the conclusions drawn. For general $F(\hat{A})$, $h(\tilde{p})$ can be written as

$$h(\tilde{p}) = \prod_{k=1}^{n} F[a_0(n - i_k(\tilde{p}))]. \quad \text{(A6)}$$

The only restriction on the function $F(x)$, apart from the required $F(0) = 1$ (see Sec. II B), is that all the $n!$ numbers $\{h(\tilde{p})\}$ appearing in the above equation be positive and distinct. Almost any function with $F(x) > 0$ for $x > 0$ will do.

APPENDIX B: $\lambda_K \neq 0$

We shall prove here that λ_k defined in Eq. (37) is nonzero for $k \neq n!$ (in Sec. III C we have remarked that $\lambda_{n!} > 0$). The proof is by contradiction. Let us assume one or more of the λ_k with $k < n!$ vanish, so that $\det H = 0$. Thus, if we interchange rows or columns of H, the determinant remains zero. Let us reorder the columns so that the upper row is composed of positive numbers in order of increasing magnitude, which we shall again denote by $h_1, h_2, \cdots \ h_{n!}$. By appropriately exchanging rows we can bring the new matrix, H', to look exactly like that in Eq. (34) with $0 < h_1 < h_2 < \cdots < h_{n!}$. Obviously $\det H' = 0$. We shall denote the eigenvalues of H' by λ_k'.

According to Eq. (35), ${\varepsilon_m}^k {\varepsilon_1}^k = {\varepsilon_{m+1}}^k$. Thus

$$\lambda_k'(1 - {\varepsilon_1}^k) = h_1 + (h_2 - h_1){\varepsilon_1}^k + (h_3 - h_2){\varepsilon_2}^k + \cdots + (h_{n!} - h_{n!-1}){\varepsilon_{n!-1}}^k - h_{n!}. \quad \text{(B1)}$$

Taking the absolute value of Eq. (B1) we find that

$$|\lambda_k'(1-{\varepsilon_1}^k)| \geq h_{n!} - \left| h_1 + (h_2-h_1){\varepsilon_1}^k + (h_3-h_2){\varepsilon_2}^k + \cdots + (h_{n!}-h_{n!-1}){\varepsilon_{n!-1}}^k \right|, \quad \text{(B2)}$$

where we have used the fact that $|x-y| \geq ||x|-|y||$ for any two complex numbers x and y. In writing Eq. (B2) we have taken into account that its r.h.s. cannot be negative since in light of the inequality $|x+y| \leq |x|+|y|$,

$$\begin{aligned} &\left| h_1 + (h_2-h_1){\varepsilon_1}^k + (h_3-h_2){\varepsilon_2}^k + \cdots + (h_{n!}-h_{n!-1}){\varepsilon_{n!-1}}^k \right| \\ &\leq h_1 + (h_2-h_1) + (h_3-h_2) + \cdots + (h_{n!}-h_{n!-1}) = h_{n!}. \end{aligned} \quad \text{(B3)}$$

We now show that the r.h.s. of Eq. (B2) cannot vanish. For if it vanished, the definitions $\alpha_1 \equiv h_1/h_{n!}$ and $\alpha_m \equiv (h_m - h_{m-1})/h_{n!}$ for $2 \leq m \leq n!$ would imply that

$$\left| \sum_{m=1}^{n!} \alpha_m {\varepsilon_m}^k \right| = 1 \quad \text{(B4)}$$

so that $\sum_{m=1}^{n!} \alpha_m {\varepsilon_m}^k = \exp(i\gamma)$ with γ real. Therefore, we would have

$$\sum_{m=1}^{n!} \alpha_m \exp\left[i \left(\frac{2\pi k}{n!} m - \gamma \right) \right] = 1. \quad \text{(B5)}$$

On the other hand, by definition all α_m are positive and

$$\sum_{m=1}^{n!} \alpha_m = 1. \quad \text{(B6)}$$

Thus Eq. (B5) can hold only if $k=0$ or $k=n!$ and $\gamma = 0 \pmod{2\pi}$. We conclude that for $k \neq 0$ and $k \neq n!$, the r.h.s. of Eq. (B2) is necessarily positive; the equation then shows that $\lambda_k' \neq 0$ for $k = 1, 2, \cdots, n!-1$. From Eq. (37) it again follows that $\lambda_{n!}' > 0$ because the $n!$-th power of all ε_m is unity. Thus, contrary to assumption, $\det H = \det H'$ cannot vanish. The contradiction tells us that all λ_m of the original matrix H are nonvanishing.

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